Negative correlation and log-concavity[*]

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Abstract

We give counterexamples and a few positive results related to several conjectures of R. Pemantle [33] and D. Wagner [37] concerning negative correlation and log-concavity properties for probability measures and relations between them. Most of the negative results have also been obtained, independently but somewhat earlier, by Borcea *et al.* [4]. We also give short proofs of a pair of results from [33] and [4]; prove that "almost exchangeable" measures satisfy the "Feder-Mihail" property, thus providing a "non-obvious" example of a class of measures for which this important property can be shown to hold; and mention some further questions.

1 Introduction

This paper is concerned with negative correlation and log-concavity properties and relations between them, with much of our motivation provided by [33] and [37]. In particular, we give counterexamples to several conjectures from these papers, and a positive answer to one question from [7]. While writing the present paper, we learned that some of the conjectures disproved here were also recently disproved in [4]. The present examples seem a little simpler and more natural, and also show a little more, so are thought to still be of interest. We also give short proofs of some of the results in [4] and [33], and show that some versions of what was for us the most interesting of the conjectures of [33] fail even for the natural example of competing urn measures, the main point here being verification of a fairly strong negative correlation property, "conditional negative association," for such measures (proof of which will appear separately). In this long introduction we summarize these developments and some of the relevant open problems.

Given a finite set S, denote by $\mathcal{M} = \mathcal{M}_S$ the set of probability measures on $\Omega = \Omega_S = \{0,1\}^S$. As a default we take $S = [n] = \{1, \ldots, n\}$ (which for us is simply a generic n-set), using Ω_n in place of $\Omega_{[n]}$. We will occasionally identify Ω with the Boolean algebra $2^{[n]}$ (the collection of subsets of $[n]$ ordered by inclusion) in the natural way (namely, identifying a set with its indicator). Recall that an event $\mathcal{A} \subseteq \Omega$ is *increasing* (really, nondecreasing) if $x \geq y \in \mathcal{A}$ implies $x \in \mathcal{A}$ (where we give Ω the product order), and similarly for *decreasing*. While our concern here is with negative dependence properties, for perspective we first recall one or two points regarding their better understood positive counterparts.

Positive correlation and association.

Recall that events $\mathcal{A}, \mathcal{B}$ in a probability space are *positively correlated*—we write $\mathcal{A} \uparrow \mathcal{B}$—if $\Pr(\mathcal{AB}) \geq \Pr(\mathcal{A}) \Pr(\mathcal{B})$. The joint distribution of random variables $X_1, ..., X_n$—here always $\{0,1\}$-valued—is said to be *positively associated* (PA) if any two events both increasing in the X_i's are positively correlated. (This is easily seen to be equivalent to the property that for any two increasing functions f, g of the X_i's one has $\mathsf{E}(fg) \geq \mathsf{E}(f)\mathsf{E}(g)$.)

The seminal result here is *Harris' Inequality* [23], which says that product measures are PA. (The special case of uniform measure on Ω was rediscovered in [28], and in combinatorial circles has often been called *Kleitman's Lemma*.) The best known—and most useful—extension of Harris' Inequality is the *FKG Inequality* of Fortuin, Kasteleyn, and Ginibre [15]:

Theorem 1 *If $\mu \in \mathcal{M}$ satisfies*

$$\mu(\eta)\mu(\tau) \leq \mu(\eta \wedge \tau)\mu(\eta \vee \tau) \qquad \forall \eta, \tau \in \Omega \tag{1}$$

(where $\wedge, \vee$ denote meet and join in the product order on Ω), then μ is PA.

(Stronger still, and also very useful, is the *Ahlswede-Daykin* or *"Four Functions" Theorem* [1], whose statement we omit.)

The *positive lattice condition* (1) (a.k.a. the *FKG lattice condition* or *log supermodularity*) is equivalent to *conditional positive association*, the property that every measure obtained from μ by conditioning on the values of some of the variables is PA; this follows easily from Theorem 1 and is a good way to make sense of (1). One also says that μ with (1) is an *FKG measure*. See, e.g., [3], [17], [29], [16], [13], [14] for a small sample of applications of these notions in combinatorics, probability, statistical mechanics, statistics and computer science.

Negative association and related properties.

While negative correlation has the obvious meaning ($\mu(\mathcal{AB}) \leq \mu(\mathcal{A})\mu(\mathcal{B})$, denoted $\mathcal{A} \downarrow \mathcal{B}$), negative *association* requires a little care (for instance, $\mathcal{A} \uparrow \mathcal{A}$ holds strictly for any $\mathcal{A}$ with $\mu(\mathcal{A}) \notin \{0,1\}$). Say $i \in [n]$ *affects* event $\mathcal{A}$ if there are $\eta \in \mathcal{A}$ and $\tau \in \Omega \setminus \mathcal{A}$ with $\eta_j = \tau_j$ for all $j \neq i$, and write $\mathcal{A} \perp \mathcal{B}$ if no coordinate affects both $\mathcal{A}$ and $\mathcal{B}$. Then $\mu \in \mathcal{M}$ is *negatively associated* (or has *negative association*; we use "NA" for either) if $\mathcal{A} \downarrow \mathcal{B}$ whenever $\mathcal{A}, \mathcal{B}$ are increasing and $\mathcal{A} \perp \mathcal{B}$. We say that μ has *negative correlations* (or *is NC*) if $\eta_i \downarrow \eta_j$ (that is, $\{\eta_i = 1\} \downarrow \{\eta_j = 1\}$) whenever $i \neq j$.

Negative association turns out to be a much subtler property than PA. Pemantle [33] proposes a number of questions regarding conditions related to NA, and possible implications among them; we sketch what we need from this, and refer to [33] for a more thorough discussion (and more motivation). The properties of interest for us are those obtained from NC and NA by requiring closure under either conditioning or imposition of external fields. We first define these operations.

Unless specified otherwise, in this paper *conditioning* always means fixing the values of some variables (and this specification is always assumed to have positive probability); thus a measure obtained from $\mu \in \mathcal{M}$ by conditioning is one of the form $\mu(\cdot | \eta_i = \xi_i \ \forall i \in I)$ for some $I \subseteq [n]$ and $\xi \in \{0,1\}^I$, which we regard as a measure on $\Omega_{[n] \setminus I}$. (If we think of Ω as $2^{[n]}$, then conditioning amounts to restricting our measure to some interval $[J, K]$ of $2^{[n]}$ (and normalizing).)

For $W = (W_1, \ldots, W_n) \in \mathbf{R}_+^n$ and $\mu \in \mathcal{M}$, define $W \circ \mu \in \mathcal{M}$ by

$$W \circ \mu(\eta) \ \propto \ \mu(\eta) \prod W_i^{\eta_i} \tag{2}$$

(meaning, as usual, that the left side is the right side multiplied by the appropriate normalizing constant). Borrowing Ising terminology, one says that $W \circ \mu$ is obtained from μ *by imposing the*

external field W (though to make this specialize correctly to the Ising model, we should really take the "field" to be h given by $h_i = \ln W_i$). It will be convenient to allow $W_i = \infty$, which we interpret as conditioning on $\{\eta_i = 1\}$; similarly we interpret $W_i = 0$ as conditioning on $\{\eta_i = 0\}$.

A third standard operation is projection: the *projection* of μ on $J \subseteq [n]$ is the measure μ' on $\{0,1\}^J$ obtained by integrating out the variables of $[n] \setminus J$; that is,

$$\mu'(\xi) = \sum \{\mu(\eta) : \eta \in \Omega, \eta_i = \xi_i \ \forall i \in J\} \quad (\xi \in \{0,1\}^J).$$

A basic motivation for much of [33] was the desire for a natural and robust notion (or notions) of negative dependence, one measure of naturalness (and also of usefulness) being invariance under some or all of the preceding operations (and a few others that we will not discuss here). This leads in particular to the following classes, which were alluded to above.

We say that $\mu \in \mathcal{M}$ is *conditionally negatively correlated* (CNC) if every measure obtained from μ by conditioning is NC, and NC+ if every measure obtained from μ by imposition of an external field is NC. *Conditional negative association* (CNA) and NA+ are defined analogously. Of course NC+ and NA+ imply CNC and CNA respectively. (This would be true even if we did not allow $W_i \in \{0, \infty\}$ in (2), since a limit of NC measures is again NC, and similarly for NA; but there are properties of interest—in particular the Feder-Mihail property below—for which things go a little more smoothly with the present convention.)

Note that Pemantle uses CNA+ where we use NA+, but it is easy to see that the two notions coincide. In general he uses "+" for closure under both projections and external fields, but for the properties we are considering, this collapses to the definitions above: it is easy to see that all of the properties NC, CNC, NC+, NA, CNA, NA+ are preserved by projections.

Following [7], [37], we will also sometimes use the term *Rayleigh* for NC+. (The reference is to Rayleigh's monotonicity law for electric networks; see the second paragraph following Conjecture 10 below or e.g. [11] or [7].) We should also say a little more about the relation between our usage and that of [33], for which we need the *negative lattice condition* (NLC) for $\mu \in \mathcal{M}$:

$$\mu(\eta)\mu(\tau) \geq \mu(\eta \wedge \tau)\mu(\eta \vee \tau) \qquad \forall \eta, \tau \in \Omega. \tag{3}$$

This is of course the analogue of (1), but turns out to be not nearly as useful, a crucial difference being that, unlike (1), it is not preserved by projections. Following [33], we say that μ has the *hereditary negative lattice condition* (h-NLC) if all projections of μ satisfy the NLC, and that μ is *h-NLC+* if every measure obtained from μ by imposition of an external field is h-NLC. It is not hard to see that there are more names here than properties:

Proposition 2 (a) *The properties CNC and h-NLC are equivalent.*

(b) *The properties NC+ and h-NLC+ are equivalent.*

This has also been observed in [4] (see their Proposition 2.2 for (b) and Remark 2.2 for a statement equivalent to (a)), so we will not prove it here, but briefly: (a) clearly implies (b); h-NLC trivially implies CNC; and the reverse implication follows easily from the observation that the support of a CNC measure is convex (i.e. $\mu(\eta), \mu(\tau) > 0$ implies $\mu(\sigma) > 0$ whenever $\eta \leq \sigma \leq \tau$), proof of which is identical to (e.g.) that of [37, Theorem 4.2].)

The next conjecture would be tremendously interesting.

Conjecture 3 ([33]) (a) *The properties CNC and CNA are equivalent.*

(b) *The properties NC+ and NA+ are equivalent.*

See [33, Conjectures 2 and 3]. Note that in each case it is enough to show that the first named property implies NA. As shown in [33], CNA does not imply NC+; we will later (Theorem 18) see a "naturally occurring" example of this. See also Conjecture 15 below for one approach to proving Conjecture 3(b).

Recall that $\mu \in \mathcal{M}$ is *exchangeable* if it is invariant under permutations of the coordinates (that is, $\mu(\eta_{\sigma(1)}, \ldots, \eta_{\sigma(n)}) = \mu(\eta_1, \ldots, \eta_n)$ for any $\eta \in \Omega$ and permutation σ of $[n]$), or, equivalently, if $\mu(\eta)$ depends only on $|\eta| := \sum \eta_i$. We say μ is *almost exchangeable* if it is invariant under permutations of some subset of $n - 1$ of the variables.

Pemantle shows [33, Theorem 2.7] that for symmetric measures the properties CNC, NC+, CNA and NA+ are equivalent, while [4] proves Conjecture 3 for almost exchangeable measures:

Theorem 4 ([4], Corollary 6.6) *For almost exchangeable measures*
(a) *the properties CNC and CNA are equivalent, and*
(b) *the properties NC+ and NA+ are equivalent.*

In Section 2 we give quick proofs of both these results. (Note that, in contrast to the exchangeable case, CNA and NC+ are not equivalent for almost exchangeable measures; see Theorem 18 and Example 32.) It may be worth noting that, despite its apparent simplicity, the class of almost exchangeable measures is considerably richer than the class of exchangeable measures; in particular, the examples proving Theorem 6 below (and also those of [4]) are almost exchangeable.

Log-concavity

Recall that a sequence $a = (a_0, \ldots, a_n)$ of real numbers is *unimodal* if there is some $k \in \{0, \ldots, n\}$ for which $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_n$, and is *log-concave* (LC) if $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq n-1$. Of course a nonnegative LC sequence with no internal zeros is unimodal. Following [33] we say that a (as above) is *ultra-log-concave* (ULC) if the sequence $(a_i/\binom{n}{i})_{i=0}^n$ is log-concave and has no internal zeros.

We also say that $\mu \in \mathcal{M}$ is ULC if its *rank sequence*, $(\mu(|\eta| = i))_{i=0}^n$, is ULC. We define "$\mu$ is LC" and "μ is unimodal" similarly, except that for the former we add the stipulation that the rank sequence has no internal zeros. (It would be convenient to also make this a requirement for "LC" for sequences, but we politely adhere to the standard definition.)

Pemantle shows [33, Theorem 2.7] that for exchangeable measures, ULC coincides with CNC, NC+, CNA and NA+. He conjectures (see his Conjecture 4) that each of the latter properties implies ULC for general μ; more precisely, this is a set of four conjectures, the weakest of which is the one with the strongest hypothesis:

Conjecture 5 *NA+ implies ULC.*

(He also conjectures that NA implies ULC, but, as noted in [4], this is easily seen to be incorrect, even for exchangeable measures.)

One of the stronger versions of Pemantle's conjecture—that the Rayleigh property (i.e. NC+) implies ULC—was separately proposed by Wagner in [37], where it was called the "Big Conjecture." (The overlap seems due to the failure in [33], [37] to notice Proposition 2(b).) We will say more about Wagner's motivation below. In Section 3 we give a family of examples that disproves all these conjectures and more:

Theorem 6 *Conjecture 5 is false; in fact NA+ does not even imply unimodality.*

As mentioned earlier, we recently learned that the first part of this was discovered a little earlier by Borcea *et al.* [4]. The present examples are slightly smaller (12 variables as opposed to 20 for violation of ULC) and simpler, and also disprove more, as the example of [4] is LC.

The examples for Theorem 6 also turn out to disprove Conjectures 8 and 9 of [33]; again, the first of these is also disproved by the example of [4]. Statements of these conjectures are deferred to Section 3.

A more particular notion than ULC, from [7], is as follows. For positive integer m, $\mu \in \mathcal{M}$ is said to have the property LC[m] if, for every $S \subseteq [n]$ of size at most m, every measure obtained from μ by imposing an external field and then projecting on S is ULC. Choe and Wagner [7, Theorem 4.6] show that the three properties NC+, LC[2], and LC[3] are equivalent. (Strictly speaking, [7] is confined to a smaller class of μ's, but the proof is valid in the present generality.) They ask whether NC+ implies LC[4]. (Of course, since projections preserve NC+, Wagner's conjecture above would say that NC+ implies LC[m] for every m.) The next result is proved in Section 4.

Theorem 7 *NC+ implies LC[5].*

Thus NC+ also implies LC[4], whereas the examples for Theorem 6 will show that

$$\text{NC+ does not imply LC[12].} \tag{4}$$

We don't know what happens between 5 and 12. Of course Theorem 7 is now less interesting than formerly, when it was thought to be a step in the direction of Conjecture 5.

Let us pause here to mention a strengthening of ULC. For $\mu \in \mathcal{M}$ set

$$\alpha_i(\mu) = \binom{n}{i}^{-1} \sum_{\eta \in \Omega, |\eta| = i} \mu(\eta)\mu(\underline{1} - \eta)$$

(where $\underline{1} = (1, \ldots, 1)$). Say that $\mu \in \mathcal{M}_{2k}$ has the *antipodal pairs property* (APP) if $\alpha_k(\mu) \geq \alpha_{k-1}(\mu)$, and that $\mu \in \mathcal{M}_n$ has the *conditional antipodal pairs property* (CAPP) if any measure obtained from μ by conditioning on the values of some $n - 2k$ variables (for some k) has the APP. (Note these properties are not affected by external fields.)

Theorem 8 *The CAPP implies ULC.*

This is reminiscent of an observation of T. Dowling [10]; see the paragraph following Conjecture 12. As pointed out to us by David Wagner [38], Theorem 4.3 of [36], which was motivated by [10], feels similar to Theorem 8. In fact, Theorem 8 implies strengthenings of Theorem 4.3 and a few other results in [36]; details of this connection will appear elsewhere [26].

Though we will not do so here, we can strengthen Theorem 7 to its antipodal pairs version (gotten by replacing ULC by CAPP in the definition of LC[m]). The proof of Theorem 8 requires some work, depending, *inter alia,* on Delsarte's inequalities [8], and will appear elsewhere [26].

Mason's Conjecture

Here we want to say a little about the motivation for Wagner's ("big") conjecture and mention a few related questions. For this discussion we regard a *matroid* as a collection $\mathcal{I}$ of independent sets, subsets of some ground set E. We will not go into matroid definitions; see e.g. [40] or [32]. Prototypes are the collection of (edge sets of) forests of a graph (with edge set E)—this is a *graphic matroid*—and (as it turns out, more generally) the collection of linearly independent subsets of

some finite subset E of some (not necessarily finite) vector space. For present purposes not too much is lost by thinking only of graphic matroids.

We are interested in the *independence numbers* of a matroid $\mathcal{I}$, that is, the numbers

$$a_k = a_k(\mathcal{I}) = |\{I \in \mathcal{I} : |I| = k\}| \quad k = 0, \ldots, n,$$

concerning which we have a celebrated conjecture of J. Mason:

Conjecture 9 ([31]) *For any matroid $\mathcal{I}$ on a ground set of size n, the sequence $a = a(\mathcal{I}) = (a_0, \ldots, a_n)$ is ULC.*

(Note that a will typically end with some zeros, and also that in the graphic case n counts edges, not vertices.) Of course one can relax Conjecture 9 by asking for LC or unimodality in place of ULC. In fact unimodality, first suggested by Welsh [39], was the original conjecture in this direction, and even this, even for graphic matroids, remains open. (See [35] or [5] for much more on log-concavity in combinatorial settings.)

From the present viewpoint, Mason's Conjecture asks for ultra-log-concavity of uniform measure on $\mathcal{I}$ (regarded in the usual way as a subset of $\{0,1\}^E$). In case $\mathcal{I}$ is graphic such a measure is a *uniform spanning forest* (USF) measure ("spanning" because we think of a member of $\mathcal{I}$ as a subgraph that includes all vertices). These measures are also very interesting from a correlation standpoint.

Conjecture 10 *USF measures are Rayleigh.*

This natural guess was perhaps first proposed in [25] (which was circulated in the combinatorial community as early as 1993, but took a while to get to press). It is also, for example, Conjecture 5.11.2 in [37]. (The statement in [25] is (in present language) that USF measures are NC, but it is not hard to see that this is equivalent.)

As essentially shown by Brooks et al. [6], the analogue of Conjecture 10 for uniform measure on the spanning *trees* of a (finite) graph amounts to Rayleigh's monotonicity law for electric networks (again, see [11]). This was extended by Feder and Mihail ([14], to which we will return shortly) to say that such measures are in fact NA+ (more precisely, this is what their proof gives).

For use below, let us call a measure obtained from a USF measure by imposition of an external field—equivalently, a measure μ on the spanning forests of some finite graph G with, for some $W : E(G) \to \mathbf{R}^+$, $\mu(F) \propto \prod_{e \in F} W(e)$—a *weighted spanning forest (weighted SF)* measure, and define *weighted spanning tree (WST)* measures and *weighted matroid measures* (replace "forest" by "independent set") similarly. (We avoid "WSF" since it means *wired* SF; see e.g. [30].)

One should note that, while the intuition for Conjecture 10 may seem clear—presence of a given edge e makes it easier for a second edge f to complete a cycle—this may be misleading, since the same intuition applies to uniform measure on the independent sets of a general matroid, for which NC need *not* hold (as can be derived from an example of Seymour and Welsh [34]). Some evidence for Conjecture 10, and its analogue for spanning connected subgraphs, is given in [20]. Also worth mentioning here—though without definitions; see [19]—is the following far-reaching extension of Conjecture 10, which has been "in the air" for a while (e.g. [33], [18]).

Conjecture 11 *Any random cluster measure with $q < 1$ is NA+.*

(Equivalently, such measures are NA.) Limiting cases include the aforementioned uniform measures on forests, spanning trees and connected subgraphs of a graph; again see [19]. Conjecture 11 with

NC+ in place of NA+ is proved for series-parallel graphs (part of a more general matroid statement) in [37] (see Example 5.1 and Theorem 5.8(d)).

Of course Wagner's "Big Conjecture," if true, would have implied Mason's Conjecture for any class of matroids for which one could establish the Rayleigh property (meaning, of course, for uniform measure on independent sets). Conjecture 10 says that graphic matroids should be such a class, and Wagner [37, Conj. 5.11] suggests a sequence of strengthenings of this. (Mason's Conjecture also partly motivated Conjecture 10 in [25], though at the time the connection was not much more than a feeling that the issues underlying the two were similar.)

Let us also mention that, as far as we know, the following strengthening of Mason's Conjecture could be true.

Conjecture 12 *Weighted matroid measures have the CAPP.*

(Of course it is enough to prove APP.) We can now fill in our earlier allusion to [10]: Dowling observed that the LC version of Mason's Conjecture would follow from the assertion that, for any matroid $\mathcal{I}$ on a ground set E of size $2k$, the number of ordered partitions $E = I \cup J$ with $I, J \in \mathcal{I}$ and $|I| = |J| = k$ is at least as large as the number with $|I| = k - 1$ (and $|J| = k + 1$), and showed this is true for $k \leq 7$. In fact, Dowling's proof also shows that, for $k \leq 5$, uniform measure on the independent sets of any matroid on a ground set of size $2k$ has the APP. This gives Conjecture 12 for matroids on ground sets of size at most 11 and, via (a slight generalization of) Theorem 8, proves that, for any matroid on a set of size n, the sequence $\{a_i\}$ of independence numbers is "ULC up to 6," meaning $(a_k/\binom{n}{k})^2 \geq (a_{k-1}/\binom{n}{k-1})(a_{k+1}/\binom{n}{k+1})$ for $k \leq 5$; see [26] for details. This is a small improvement on the best that seems to have been known previously, namely that $\{a_i\}$ is ULC up to 4, which was shown by Hamidoune and Salaün [21].

Feder-Mihail

Say $\mu \in \mathcal{M}$ has the *Feder-Mihail property* (or *is FM*) if

$$\text{for any increasing } \mathcal{A} \subseteq \Omega, \ \{\eta_i = 1\} \uparrow \mathcal{A} \text{ for some } i \in [n],$$

and extend this to CFM and FM+ in the usual way. (Of course FM+ trivially implies CFM, but note that this implication requires that we explicitly include conditioning in our definition of "+" (i.e. we allow $W_i \in \{0, \infty\}$ in (2)), since e.g. there are situations where $W \circ \mu$ is FM for all W with positive entries but $\mu(\cdot | \eta_1 = 1)$ is not FM.) The following simple but powerful observation is essentially from [14], though given there only in a special case.

Theorem 13 (a) *If $\mu \in \mathcal{M}$ is both CNC and CFM then it is CNA.*

(b) *If $\mu \in \mathcal{M}$ is both NC+ and FM+ then it is NA+.*

(A statement equivalent to (a) is proved in [33, Theorem 1.3], and (b) follows easily from (a).)

Given the power of Theorem 13, it would be useful to identify situations where the FM property holds. This is trivially the case for μ concentrated on a level (that is, $\{\eta \in \Omega : |\eta| = k\}$ for some k; see e.g. [12, Corollary 3.2]), e.g. for WST measures (a key to [14]), and it is fairly easy to show that exchangeable measures and, more generally, "rescalings" of product measures, satisfy the stronger "normalized matching property" (NMP; see Section 2 for definitions). But in general FM seems hard to establish, and indeed we are not aware of any interesting classes of non-NMP measures that are known to be FM. Thus the following result, which is proved in Section 2, may be of some interest.

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Theorem 14 *Almost exchangeable measures are FM+.*

Note that this combined with Theorem 13 gives Theorem 4. (This is not quite the "quick" proof of Theorem 4 promised earlier, since Theorem 14 requires some effort; but, as observed in Section 2, FM (resp. FM+) for almost exchangeable measures that are also NC (resp. NC+) is much easier.)

Despite the (apparent) difficulty of proving FM, the property seems to tend to hold for measures not deliberately constructed to violate it. In particular we would like to propose, perhaps a bit optimistically, the following possibilities.

Conjecture 15 *The Feder-Mihail property holds for*
(a) *Rayleigh measures,*
(b) *weighted SF measures, and (more generally)*
(c) *weighted matroid measures.*

Note that, in view of Theorem 13, (a) would imply Conjecture 3(b) (the corresponding approach to Conjecture 3(a) fails because CNC measures need not be FM), while (b) together with Conjecture 10 would say that USF measures are NA+. (Extending this to matroids via (c) fails because Conjecture 10 does.)

Competing urns

One of the principal motivating examples for [33] is "competing urns." Here we have m balls which are thrown independently into urns $1, \ldots, n$ according to some (common) distribution, let X_i be the indicator for occupation of urn i, and consider the corresponding measure μ on Ω (that is, the law of $(X_1, \ldots, X_n)$). (Formally we may take a random $\sigma : [m] \to [n]$ with the $\sigma(i)$'s i.i.d., and $X_i = \mathbf{1}_{\{\sigma^{-1}(i) \neq \emptyset\}}$.) Call a μ of this type a *competing urn measure*.

The competing urns model is explored in some detail by Dubhashi and Ranjan [13], who in particular prove that competing urn measures are NA. Another proof of this is given in [33].

More generally one may consider thresholds $a_1, \ldots, a_n$, with X_i the indicator of $\{|\sigma^{-1}(i)| \geq a_i\}$—for lack of a better name, we then call the law of $(X_1, \ldots, X_n)$ an *extended* competing urn measure—and the arguments of [13] and [33] apply to show that such measures are again NA. (Actually [13] proves the stronger statement that the (law of the) random variables $X_i(j) = \mathbf{1}_{\{\sigma(i)=j\}}$ is NA.)

Here again, we have a suggestion from [33]:

Conjecture 16 *Any competing urn measure is ULC.*

In fact Pemantle conjectures something more general that we will not state, since, unfortunately, Conjecture 16 is not true.

Proposition 17 *Competing urn measures need not be LC.*

(See Example 31.) It is also not hard to give examples of non-Rayleigh competing urn measures (Example 32). On the other hand, one may argue that in the context of competing urns, external fields are less natural than conditioning; at any rate, it does turn out that even extended competing urn measures are nicely behaved under conditioning:

Theorem 18 *Extended competing urn measures are CNA.*

Thus, as mentioned earlier, we have a natural class of measures for which CNA does not imply Rayleigh, and, combining with Proposition 17, a (natural) counterexample to the strengthening of Conjecture 5 obtained by replacing NA+ by CNA (this is again one of the versions of Conjecture 4 of [33]).

While Theorem 18 seems like it ought to be easy, we do not know a simple argument, even in the ordinary (non-extended) case, and, to keep this paper from getting too long, will give the proof elsewhere [27]. That it might not be completely straightforward is suggested by our inability to decide whether it is still true (again, even in the ordinary case) if we drop the requirement that the balls be identically distributed. The aforementioned NA for $X_i(j)$'s proved in [13] is true in this generality, which gives NA for these more general urn measures. (The argument of [33] does not work with nonidentical balls.)

2 Exchangeable and almost exchangeable measures

We need a few more definitions. We extend the definitions of exchangeable and almost exchangeable measures (given following Conjecture 3) to general functions on Ω in the obvious way ($f : \Omega \to \mathbf{R}$ is *almost exchangeable* if it is invariant under permutations of some subset of $n-1$ of the variables and *exchangeable* if it is invariant under permutations of all the variables). We also extend our notation for positive and negative correlation to functions: for $f, g : \Omega \to \mathbf{R}$, we write $f \uparrow g$ if $\mathsf{E}(fg) \geq \mathsf{E}(f)\mathsf{E}(g)$ (and similarly for $f \downarrow g$); we will also write, e.g., $\mathcal{A} \uparrow f$ for $\mathbf{1}_{\mathcal{A}} \uparrow f$. A stronger statement is that $\mathcal{A}$ is *stochastically increasing in f*, that is, that $\Pr(\mathcal{A}|f = t)$ is increasing in t, where we restrict to values of t for which $\Pr(f = t)$ is positive. (Note that our use of the notation $\mathcal{A} \uparrow f$ differs from that in [33].) Following [12], for a function $f : \Omega \to \mathbf{R}$ and a measure $\mu \in \mathcal{M}$, we say that $i \in [n]$ is a *variable of positive influence* for the pair (f, μ) (or $(\mathcal{A}, \mu)$ if $f = \mathbf{1}_{\mathcal{A}}$) if $\eta_i \uparrow f$. Thus the FM property for μ says that for every increasing $\mathcal{A}$ there is a variable of positive influence for $(\mathcal{A}, \mu)$. In [12], Dubhashi *et al.* prove the (easy) result that (f, μ) has a variable of positive influence if f is increasing and at least one of f, μ is exchangeable, and ask for other classes of function-measure pairs having variables of positive influence. Here we prove the following result, which, as we will see shortly, implies Theorem 14.

Theorem 19 *If there is a variable l for which*

$$\mathsf{E}[f|\eta_l = j, \sum_{i \neq l} \eta_i = k] \text{ is increasing in } j \text{ and } k \tag{5}$$

(for pairs (j, k) for which the conditioning event has positive probability), then (f, μ) has a variable of positive influence. In particular, if f is increasing and almost exchangeable then (f, μ) has a variable of positive influence.

As observed earlier, Theorem 4 is an immediate consequence of Theorems 13 and 14; but it does not require the full strength of Theorem 14, and before proving Theorem 19 we will give an easier argument, together with a quick proof of the following result from [33].

Theorem 20 *For exchangeable measures the properties CNC, CNA, NC+, NA+, and ULC are equivalent.*

For these arguments and the derivation of Theorem 14 we need a little background. We first recall Chebyshev's Inequality, here stated in our terminology.

Proposition 21 *Any probability measure on a totally ordered set is PA.*

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(Where, of course, PA is as for measures on $\{0,1\}^S$: $f \uparrow g$ for any two increasing functions f, g.)

Recall that for probability measures μ and ν, μ *stochastically dominates* ν (written $\mu \succeq \nu$) if $\mu(\mathcal{A}) \geq \nu(\mathcal{A})$ for every increasing $\mathcal{A}$. Writing μ_k for the conditional measure $\mu(\cdot | \sum \eta_i = k)$ (defined only when $\mu(\sum \eta_i = k) > 0$), we say that μ has the *normalized matching property* (NMP) if $\mu_l \succeq \mu_k$ whenever $l \geq k$ and both conditional measures are defined. (This generalizes the usual definition, for which see e.g. [2].) The NMP is equivalent to the property that every increasing event $\mathcal{A}$ is stochastically increasing in $\sum \eta_i$, which implies (easily and directly, by Proposition 21, or essentially by Proposition 1.2 in [33]) that $\mathcal{A} \uparrow \sum \eta_i$, and thus (since expectation is linear) that $\mathcal{A} \uparrow \eta_i$ for some i. This discussion gives the next observation.

Proposition 22 *The NMP implies FM.*

Conjecture 8 of [33] says that NA+ implies NMP, but this is false; see Conjecture 26 and Theorem 28 in Section 3. Given $\mu \in \mathcal{M}$ and nonnegative sequence $a = (a_i)_{i=0}^n$, the *generalized rank rescaling of μ by a* is the measure $a \otimes \mu \in \mathcal{M}$ with

$$a \otimes \mu(\eta) \propto a_{|\eta|} \mu(\eta).$$

(To be precise, we only make this definition when the right side is not identically zero.) This generalizes the rank rescaling operation in [33], which required that a be LC with no internal zeros. Observe that (since $\mu_k = (a \otimes \mu)_k$ whenever $a_k > 0$) generalized rank rescalings preserve the NMP.

Lemma 23 *Product measures (and, consequently, generalized rank rescalings of product measures) have the NMP.*

(A proof is sketched in [12, Section 4.2]. More generally, any product of LC measures with the NMP again has these properties; this was proved by Harper [22], and can also essentially be gotten from a combinatorial version proved in [24], [2].) For the proof of Theorem 4 we also need the following standard observation, an easy consequence of Proposition 21.

Lemma 24 *Let $f, g : \Omega \to \mathbf{R}$, and suppose for some event $\mathcal{B}$*
(i) *each of f, g is positively correlated with $\mathcal{B}$, and*
(ii) *f and g are conditionally positively correlated given each of $\mathcal{B}$, $\Omega \setminus \mathcal{B}$.*
Then f and g are positively correlated.

Proof of Theorem 14. Let $\mu' \in \mathcal{M}$ be invariant under permutations of the variables $1, \ldots, n-1$, and $\mu = W \circ \mu'$ for some $W \in \mathbf{R}_+^n$. We may assume all W_i are finite and strictly positive, since otherwise we can reduce the number of variables (note that any measure gotten from an almost exchangeable measure by conditioning is again almost exchangeable). Then, with $\nu \in \mathcal{M}_{n-1}$ the product measure satisfying

$$\nu(\eta) \propto \prod_{i \in [n-1]} W_i^{\eta_i},$$

we have

$$\mu(\cdot | \eta_n = 0, \sum_{i \in [n-1]} \eta_i = k) = \mu(\cdot | \eta_n = 1, \sum_{i \in [n-1]} \eta_i = k) = \nu_k,$$

so (by Lemma 23) $f = \mathbf{1}_{\mathcal{A}}$ satisfies (5) with $l = n$ for every increasing event $\mathcal{A}$.

$\blacksquare$

Proof of Theorem 4. By Theorem 13, it suffices to show that *every NC measure that can be gotten by applying an external field to an almost exchangeable measure is FM.* Let μ be such a measure, obtained by imposing an external field on a measure invariant under permutations of coordinates $\{2,\ldots,n\}$, and let $\mathcal{A}$ be an increasing event. We should show that $\mathcal{A} \uparrow \eta_i$ for some $i \in [n]$. We may assume that all coordinates of the external field are finite and strictly positive (or we can reduce the number of variables), and that $\mathcal{A} \downarrow \eta_1$ (or we are done). For $j \in \{0,1\}$, the conditional measure $\mu(\cdot|\eta_1 = j)$ is a generalized rank rescaling of a product measure, so by Lemma 23 and the discussion in the paragraph preceding Proposition 22 we have $\mathcal{A} \uparrow f := \sum_{i\neq 1} \eta_i$ conditionally given either of the events $\{\eta_1 = 0\}$, $\{\eta_1 = 1\}$. Since μ is NC, we have $\eta_1 \downarrow \eta_i$ for all $i \in \{2,\ldots,n\}$, so that $\eta_1 \downarrow f$. But then applying Lemma 24 with $g = \mathbf{1}_{\mathcal{A}}$ and $\mathcal{B} = \{\eta_1 = 0\}$ gives $\mathcal{A} \uparrow f$, whence $\mathcal{A} \uparrow \eta_i$ for some $i \in \{2,\ldots,n\}$.

∎

Proof of Theorem 20. It suffices to show that CNC implies ULC, and that ULC implies NA+. The first implication is easy: CNC implies NLC (cf. Proposition 2), which for exchangeable measures is equivalent to ULC. For the second implication, our main point is that we can eliminate much of the work in [33] by observing that exchangeable measures are FM+ (by Lemma 23 and Proposition 22; of course this is also an instance of Theorem 14, but not one that requires the less trivial Theorem 19), so that by Theorem 13(b) it is enough to show that ULC implies NC+. This is a special case of the observation, proved in Lemma 2.8 of [33], that a measure obtained from an exchangeable ULC measure by imposing an external field that is identically 1 on $J \subseteq [n]$, followed by projection on J, is exchangeable and ULC.

∎

Proof of Theorem 19. First observe that if $f : \Omega_n \to \mathbf{R}$ is increasing and invariant under permutations of coordinates in $[n] \setminus \{l\}$ then (5) is satisfied for every $\mu \in \mathcal{M}$, so the last part of Theorem 19 follows from the first.

To prove the first part, suppose, without loss of generality, that $\mu \in \mathcal{M}$ and $f : \Omega_n \to \mathbf{R}$ satisfy (5) with $l = 1$, and set $h(\eta) = \sum_{i\neq 1} \eta_i$ $(\eta \in \Omega_n)$. It suffices to show that either

$$f \uparrow \eta_1 \quad \text{or} \quad f \uparrow h. \tag{6}$$

For $i \in \{0,\ldots,n-1\}$, let

$$\alpha_i = \mu(h = i, \eta_1 = 1) \quad \text{and} \quad \beta_i = \mu(h = i, \eta_1 = 0).$$

Choose increasing, nonnegative sequences $\gamma = (\gamma_0,\ldots,\gamma_{n-1})$ and $\delta = (\delta_0,\ldots,\delta_{n-1})$ such that

$$\gamma_i = \mathsf{E}[f|h = i, \eta_1 = 1] \quad \text{and} \quad \delta_i = \mathsf{E}[f|h = i, \eta_1 = 0] \tag{7}$$

whenever the conditioning events have positive probability and $\gamma_i \geq \delta_j$ whenever $i \geq j$. (Existence of γ, δ is guaranteed by (5). This extension to values not given by (7) is convenient, but not really necessary, as these values play no role; see (8) and Lemma 25.)

Assume $f \downarrow \eta_1$, i.e.,

$$\frac{\sum \alpha_i \gamma_i}{\sum \alpha_i} \leq \frac{\sum \beta_i \delta_i}{\sum \beta_i}$$

(all sums in this proof are over $\{0, \ldots, n-1\}$ unless otherwise specified). We want to show $\mathsf{E}(fh) \geq \mathsf{E}(f)\mathsf{E}(h)$, that is,

$$\sum (\alpha_i \gamma_i + \beta_i \delta_i) \sum i(\alpha_i + \beta_i) \leq \sum i(\alpha_i \gamma_i + \beta_i \delta_i) \sum (\alpha_i + \beta_i). \tag{8}$$

(Of course the last sum is 1.) This will follow from

$$\sum \alpha_i \gamma_i \sum i\alpha_i \leq \sum i\alpha_i \gamma_i \sum \alpha_i,$$

$$\sum \beta_i \delta_i \sum i\beta_i \leq \sum i\beta_i \delta_i \sum \beta_i,$$

and

$$\sum i\alpha_i \sum \beta_i \delta_i + \sum i\beta_i \sum \alpha_i \gamma_i \leq \sum i\alpha_i \gamma_i \sum \beta_i + \sum i\beta_i \delta_i \sum \alpha_i. \tag{9}$$

The first two of these are instances of Proposition 21 (since γ and δ are increasing), so it suffices to prove the following.

Lemma 25 *Let $\alpha = (\alpha_i)_{i=0}^{n-1}$, $\beta = (\beta_i)_{i=0}^{n-1}$, $\gamma = (\gamma_i)_{i=0}^{n-1}$, and $\delta = (\delta_i)_{i=0}^{n-1}$ be nonnegative sequences (with neither of α, β identically zero). If γ and δ are increasing, $\gamma_i \geq \delta_j$ whenever $i \geq j$, and $(\sum \alpha_i \gamma_i)/(\sum \alpha_i) \leq (\sum \beta_i \delta_i)/(\sum \beta_i)$, then (9) holds.*

Proof. Since scaling α, β affects neither our hypotheses nor (9), we may assume $\sum \alpha_i = \sum \beta_i$. It suffices to show

$$\sum_{i \geq s} \alpha_i \sum \beta_i \delta_i + \sum_{i \geq s} \beta_i \sum \alpha_i \gamma_i \leq \sum_{i \geq s} \alpha_i \gamma_i \sum \beta_i + \sum_{i \geq s} \beta_i \delta_i \sum \alpha_i \tag{10}$$

for $s \in [n-1]$ (since summing (10) over s yields (9)).

Fix $s \in [n-1]$. Obviously,

$$\text{if (10) is true, then it remains true when any } \delta_i \text{ with } i \geq s \text{ is increased.} \tag{11}$$

We define $\delta' = (\delta_i')_{i=0}^{n-1}$ by

$$\delta_i' = \begin{cases} \delta_i & \text{if } i < s \\ \delta_s & \text{if } i \geq s \end{cases}$$

and consider two cases.

<u>Case 1</u>: $\sum \alpha_i \gamma_i > \sum \beta_i \delta_i'$. Then there is an increasing sequence $\delta'' = (\delta_i'')_{i=0}^{n-1}$ with $\delta_i' \leq \delta_i'' \leq \delta_i$ for all i and $\sum \alpha_i \gamma_i = \sum \beta_i \delta_i''$ (note $\sum \alpha_i \gamma_i \leq \sum \beta_i \delta_i$, since we normalized to $\sum \alpha_i = \sum \beta_i$). Since γ and δ'' are increasing, we have

$$\sum_{i \geq s} \alpha_i \sum \alpha_i \gamma_i \leq \sum_{i \geq s} \alpha_i \gamma_i \sum \alpha_i$$

and

$$\sum_{i \geq s} \beta_i \sum \beta_i \delta_i'' \leq \sum_{i \geq s} \beta_i \delta_i'' \sum \beta_i.$$

This yields (10) with δ replaced by δ'', and then (10) (for δ) follows from (11).

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<u>Case 2</u>: $\sum \alpha_i \gamma_i \leq \sum \beta_i \delta_i'$. By (11), it suffices to prove (10) with δ replaced by δ'; this is a straight-forward computation:

$$
\begin{aligned}
\sum_{i\geq s}\alpha_i \sum \beta_i \delta_i' + \sum_{i\geq s}\beta_i \sum \alpha_i \gamma_i &\leq \sum_{i\geq s}(\alpha_i + \beta_i) \sum \beta_j \delta_j' \\
&= \sum_{i\geq s}\sum_j (\alpha_i \beta_j \delta_j' + \beta_i \beta_j \delta_j') \\
&\leq \sum_{i\geq s}\sum_j (\alpha_i \beta_j \delta_i' + \beta_i \beta_j \delta_i') \\
&= \sum_{i\geq s}(\alpha_i \delta_i' + \beta_i \delta_i') \sum \beta_j \\
&\leq \sum_{i\geq s}\alpha_i \gamma_i \sum \beta_i + \sum_{i\geq s}\beta_i \delta_i' \sum \alpha_i,
\end{aligned}
$$

where we used: $\sum \alpha_i \gamma_i \leq \sum \beta_i \delta_i'$ for the first inequality; $\delta_i' \geq \delta_j'$ $\forall i \geq s$ (and $\forall j$) for the second; and $\gamma_i \geq \delta_i'$ $\forall i$ for the third. This completes the proofs of Lemma 25 and Theorem 19.

∎

3   Counterexamples

Here we give the construction for Theorem 6. As mentioned earlier, two further conjectures from [33] turn out to be disproved by the same examples, and we begin by stating these.

Conjecture 26 ([33], Conjecture 8) *NA+ implies the NMP.*

(See Section 2 for NMP.)

Recall that for $\eta, \zeta \in \Omega$, η *covers* ζ ($\eta \gtrdot \zeta$) if there is an $i \in [n]$ for which $\eta_i = 1$, $\zeta_i = 0$ and $\eta_j = \zeta_j$ $\forall j \neq i$. Following [33] we say that $\mu \in \mathcal{M}$ *stochastically covers* $\nu \in \mathcal{M}$ ($\mu \succ \nu$) if we can couple random variables η, ζ having laws μ and ν so that with probability 1, $\eta = \zeta$ or $\eta \gtrdot \zeta$; and that μ has the *stochastic covering property* (SCP) if $\mu(\cdot|\eta_i = 0) \succ \mu(\cdot|\eta_i = 1)$ for every i (where, again, we regard these as measures on $\Omega_{[n]\setminus\{i\}}$). Observe that if μ is NA+ then $\mu(\cdot|\eta_i = 0) \succeq \mu(\cdot|\eta_i = 1)$; the following strengthening was suggested by Pemantle.

Conjecture 27 ([33], Conjecture 9) *NA+ implies the SCP.*

As already mentioned, our examples invalidate both these conjectures.

Theorem 28 *Conjectures 26 and 27 are false.*

Conjecture 26 was also disproved in [4]; see also the note at the end of this section.

We now describe the examples. For a positive integer $k \geq 2$ and $\beta \in (0,1)$, let $\nu^{k,\beta}$ be the measure on Ω_{2k} with

$$
\nu^{k,\beta}(\eta) \propto \begin{cases}
1 & \text{if} \quad [|\eta| = k-1 \text{ and } \eta_1 = 1] \\
\beta^2 & \text{if} \quad [|\eta| = k-1 \text{ and } \eta_1 = 0] \\
\beta & \text{if} \quad |\eta| = k \\
\beta^2 & \text{if} \quad [|\eta| = k+1 \text{ and } \eta_1 = 1] \\
1 & \text{if} \quad [|\eta| = k+1 \text{ and } \eta_1 = 0] \\
0 & \text{otherwise}
\end{cases}
$$

Note that (clearly) $\nu^{k,\beta}$ is almost exchangeable.

Proposition 29 *The measure $\nu^{k,\beta}$ satisfies:*
(a) *NA+ if and only if $\beta \geq \frac{1}{\sqrt{2}}$*
(b) *ULC if and only if $\beta \geq 1 - \frac{2}{k+1}$*
(c) *unimodality (and also LC) if and only if $\beta \geq 1 - \sqrt{\frac{2}{k+1}}$*
(d) *NMP if and only if $\beta \geq \sqrt{1 - \frac{2}{k+1}}$*
(e) *SCP if and only if $\beta \geq \sqrt{1 - \frac{2}{k+1}}$*

For example, for $\beta = 0.71$: $\nu^{6,\beta}$ is NA+ but not ULC, giving the first part of Theorem 6 (i.e. disproving Conjecture 5); $\nu^{23,\beta}$ is NA+ but not unimodal (proving Theorem 6); and $\nu^{4,\beta}$ is NA+ but not NMP or SCP (proving Theorem 28).

Proof. We will mainly prove what we need for Theorems 6 and 28, namely "if" in (a) and "only if" in (b)-(e). The other direction in (b),(c) will come for free, but we omit the (not very difficult) verifications of the remaining implications.

Fix k and β, write ν for $\nu^{k,\beta}$, and set $r_i = \nu(|\eta| = i)$. We have, for some C,

$$r_k = C\beta\binom{2k}{k} \quad \text{and} \quad r_{k-1} = r_{k+1} = C\Big(\frac{k-1}{2k} + \frac{k+1}{2k}\beta^2\Big)\binom{2k}{k-1}.$$

Unimodality and LC for ν are equivalent to (each other and) $r_k \geq r_{k-1}$, which reduces to

$$\beta^2 - 2\beta + \frac{k-1}{k+1} \leq 0,$$

giving (c). ULC for ν is equivalent to $k^2 r_k^2 \geq (k+1)^2 r_{k-1}r_{k+1}$, which reduces to

$$(k+1)\beta^2 - 2k\beta + (k-1) \leq 0,$$

giving (b). The NMP requires that

$$\nu(\eta_1 = 1 | |\eta| = k) = \frac{1}{2}$$

be at least as large as

$$\nu(\eta_1 = 1 | |\eta| = k-1) = \frac{k-1}{(k-1)+(k+1)\beta^2},$$

from which the forward direction of (d) follows. The SCP requires

$$\nu(|\eta| = k-1 | \eta_1 = 1) \leq \nu(|\eta| = k-1 | \eta_1 = 0),$$

which reduces to

$$\binom{2k-1}{k-2} \leq \beta^2 \binom{2k-1}{k-1}$$

and yields the forward direction of (e).

It remains to prove the backward direction of (a); that is, we assume $\beta \geq 1/\sqrt{2}$ and should show ν is NA+. Since ν is almost exchangeable, Theorem 4 says we only need to show NC+, which, by symmetry, will follow if we show $\eta_1 \downarrow \eta_2$ and $\eta_2 \downarrow \eta_3$ with respect to $W \circ \nu$, for any external field W. (Our original proof of this has been shortened using some ideas from [4].) Observe that, since

a limit of NC measures is NC, it suffices to consider the case when all entries of W are finite and strictly positive.

Let $W' = (W_1, 1, \ldots, 1)$, and let ν' be the projection of $W' \circ \nu$ on $\Omega_{\{2,\ldots,2k\}}$. To prove $\eta_2 \downarrow \eta_3$ for $W \circ \nu$, it suffices to show ν' is NC+, which, since ν' is exchangeable, will follow *via* Theorem 20 if we show ν' has a ULC rank sequence. The nonzero part of the normalized rank sequence ($a_i := \nu'(|\eta| = i)/\binom{2k-1}{i})_{i=0}^{2k-1}$ is $(a_{k-2}, \ldots, a_{k+1}) \propto (W_1, W_1\beta + \beta^2, W_1\beta^2 + \beta, 1)$, which a straightforward computation shows to be LC when $\beta \geq 1/\sqrt{2}$.

That $\eta_1 \downarrow \eta_2$ for $W \circ \nu$ will follow immediately from

$$W \circ \nu(\cdot | \eta_1 = 0) \succeq W \circ \nu(\cdot | \eta_1 = 1). \tag{12}$$

Set

$$
\begin{aligned}
\pi_1 &= W \circ \nu(\cdot | \eta_1 = 0, |\eta| = k+1), \\
\pi_2 &= W \circ \nu(\cdot | \eta_1 = 0, |\eta| \in \{k-1, k\}), \\
\pi_3 &= W \circ \nu(\cdot | \eta_1 = 1, |\eta| \in \{k, k+1\}), \text{ and} \\
\pi_4 &= W \circ \nu(\cdot | \eta_1 = 1, |\eta| = k-1).
\end{aligned}
$$

It follows readily from Lemma 23 (since $\beta < 1$ and the two measures appearing in (12) are rank rescalings of a common product measure, namely the measure $\mu \in \mathcal{M}_{\{2,\ldots,2k\}}$ with $\mu(\tau) \propto \prod W_i^{\tau_i}$) that each of π_1, π_2 stochastically dominates each of π_3, π_4. Consequently, every convex combination of π_1 and π_2 stochastically dominates every convex combination of π_3 and π_4, which in particular gives (12).

■

Before closing this section, let us just mention that a more natural class of counterexamples to Conjecture 26 is *probably* provided by the following simple construction, which, as far as we know, first appeared in [9]. Given k, let G be the graph with $V(G) = \{x, y, z_1, \ldots, z_k\}$ and $E(G) = \{xy, xz_1, \ldots, xz_k, yz_1, \ldots, yz_k\}$. It is well known and easy to see (consider the event $\eta_{xy} = 1$) that for $k \geq 5$, the USF measure for G fails the NMP, so is a counterexample to Conjecture 26 if the USF measure for G is NA+. The latter would follow from Conjecture 15(b) (USF measures are FM+) for G, since Conjecture 10 for these graphs is contained in the result from [37] mentioned following Conjecture 11. (We can prove FM+ for $k \leq 5$, and even this is not so easy).

4 Proof of Theorem 7

The main point here is the following lemma, stating that NC+ implies the APP (defined before Theorem 8) for measures in $\mathcal{M}_4$. For simplicity, we set $\alpha_X = \mu(X)$ for $X \subseteq [n]$ (where we now treat Ω_n as $2^{[n]}$), often omit commas and set braces in subscripts (e.g. $\alpha_{134} = \mu(\{1, 3, 4\})$), and write α_0 for $\mu(\emptyset)$. Let $\Sigma_{r,s}^t = \sum \alpha_X \alpha_Y$, with the sum over unordered pairs $\{X, Y\}$ of subsets of $[n]$ with $|X| = r$, $|Y| = s$, and $|X \cap Y| = t$.

Lemma 30 *If* $\mu \in \mathcal{M}_4$ *is NC+, then*

$$3\Sigma_{1,3}^0 \leq 4\Sigma_{2,2}^0. \tag{13}$$

We first prove this and then give the easy derivation of Theorem 7. (Notice that we could also get Theorem 7 from Lemma 30 via Theorem 8, (13) being the only part of the CAPP that is not immediate from NC+; but, as mentioned earlier, Theorem 8 is relatively difficult, so we prefer a direct proof of Theorem 7 here.)

For convenience, we now work with unnormalized (nonnegative) measures on Ω, and say that such a measure μ with $\mu(\Omega) > 0$ has a property (CNC, NC+, etc.) iff its normalization μ' (given by $\mu'(\eta) = \mu(\eta)/\mu(\Omega)$) does. Observe that $\mathcal{A} \downarrow \mathcal{B}$ under μ if and only if

$$\mu(\mathcal{A}\bar{\mathcal{B}})\mu(\bar{\mathcal{A}}\mathcal{B}) \geq \mu(\mathcal{A}\mathcal{B})\mu(\bar{\mathcal{A}}\bar{\mathcal{B}}) \tag{14}$$

(where $\bar{\mathcal{A}} = \Omega \setminus \mathcal{A}$).

Proof of Lemma 30. Let $A = \Sigma_{2,2}^0$ ($= \alpha_{12}\alpha_{34} + \alpha_{13}\alpha_{24} + \alpha_{14}\alpha_{23}$). We may assume $\alpha_0 = \alpha_{1234} = 0$, since decreasing α_0 or α_{1234} preserves NC+ and has no effect on (13). Furthermore, by renaming variables, applying a constant external field, and scaling (none of which affect (13)), we may assume

$$\alpha_{123} = \alpha_4 = 1 \quad \text{and} \quad \alpha_1\alpha_{234}, \alpha_2\alpha_{134}, \alpha_3\alpha_{124} \leq 1. \tag{15}$$

(First, rename coordinates so $\alpha_i\alpha_{[4]\setminus\{i\}}$ is largest for $i = 4$ (and observe we may assume this largest value is strictly positive). Second, impose a uniform external field (W, W, W, W) to get $\alpha_{123} = \alpha_4$. Third, divide all values α_X by α_{123}.)

Since μ is NC+ (so in particular CNC), (15) gives $\alpha_1 \leq \alpha_{12}\alpha_{13}$ (and similarly for α_2, α_3) and $\alpha_{124} \leq \alpha_{14}\alpha_{24}$ (and similarly for α_{134}, α_{234}). It thus suffices to show

$$3(1 + xy + xz + yz) \leq 4A, \tag{16}$$

where

$$x = \alpha_{12}\alpha_{34}, \ y = \alpha_{13}\alpha_{24}, \ \text{and} \ z = \alpha_{14}\alpha_{23} \ \ (\text{so } A = x + y + z).$$

For fixed A, the left hand side of (16) is maximized when $x = y = z$; thus (16) holds whenever $A \in [1, 3]$ (as can be seen by examining the quadratic polynomial $A^2 - 4A + 3$). In view of (15) we can assume $A \leq 3$, so we just need $A \geq 1$.

Assume, for a contradiction, that $A < 1$. Negative correlation of η_2 and η_3 for the measure $(0, 1, 1, W) \circ \mu$ implies (use (14))

$$P^1(W) := (\alpha_{24}\alpha_{34} - \alpha_{234})W^2 + (\alpha_2\alpha_{34} + \alpha_3\alpha_{24} - \alpha_{23})W + \alpha_2\alpha_3 \geq 0 \quad \text{for } W > 0.$$

Similarly, negative correlation of η_1 and η_2 for $(\infty, 1, 1, W) \circ \mu$ implies

$$P_1(W) := \alpha_{124}\alpha_{134}W^2 + (\alpha_{12}\alpha_{134} + \alpha_{13}\alpha_{124} - \alpha_{14})W + (\alpha_{12}\alpha_{13} - \alpha_1) \geq 0 \quad \text{for } W > 0.$$

Similarly (interchanging 1 with either 2 or 3) we have, again for $W > 0$,

$$P^2(W) := (\alpha_{14}\alpha_{34} - \alpha_{134})W^2 + (\alpha_1\alpha_{34} + \alpha_3\alpha_{14} - \alpha_{13})W + \alpha_1\alpha_3 \geq 0,$$

$$P_2(W) := \alpha_{124}\alpha_{234}W^2 + (\alpha_{12}\alpha_{234} + \alpha_{23}\alpha_{124} - \alpha_{24})W + (\alpha_{12}\alpha_{23} - \alpha_2) \geq 0,$$

$$P^3(W) := (\alpha_{14}\alpha_{24} - \alpha_{124})W^2 + (\alpha_1\alpha_{24} + \alpha_2\alpha_{14} - \alpha_{12})W + \alpha_1\alpha_2 \geq 0, \ \text{and}$$

$$P_3(W) := \alpha_{134}\alpha_{234}W^2 + (\alpha_{13}\alpha_{234} + \alpha_{23}\alpha_{134} - \alpha_{34})W + (\alpha_{13}\alpha_{23} - \alpha_3) \geq 0.$$

We pause to show

$$\alpha_X > 0 \text{ for } X \neq \emptyset, [4]. \tag{17}$$

First we show $\alpha_X > 0$ if $|X| = 2$. Suppose for example that $\alpha_{12} = 0$. Since $P^1(W) \geq 0$ for all $W > 0$ and $\alpha_2 = 0$ (since $\alpha_2 \leq \alpha_{12}\alpha_{23}$) the coefficient of W in P^1 must be nonnegative, and thus (using $\alpha_3 \leq \alpha_{13}\alpha_{23}$) $y\alpha_{23} \geq \alpha_{23}$. If $\alpha_{23} > 0$, this gives $A \geq y \geq 1$; thus (since we are assuming $A < 1$) $\alpha_{23} = 0$, and similar reasoning shows $\alpha_1 = \alpha_3 = \alpha_{13} = 0$. Hence, $\alpha_X = 0$ unless $X = \{1, 2, 3\}$ or $4 \in X$; but then nonnegativity of P_1, P_2, and P_3 gives $\alpha_{14} = \alpha_{24} = \alpha_{34} = 0$. Thus $\alpha_X > 0$ if and only if $X = \{1, 2, 3\}$ or $X = \{4\}$; but for any such measure η_1 and η_2 are strictly positively correlated. This contradiction shows $\alpha_{12} > 0$, and similar arguments (or symmetry) give $\alpha_X > 0$ whenever $|X| = 2$.

If $\alpha_2 = 0$, then nonnegativity of the linear term in P^1 gives, as in the preceding paragraph (and using $\alpha_{23} > 0$), $A \geq 1$; thus $\alpha_2 > 0$. Similar arguments (or, again, symmetry) show α_1, α_3, α_{124}, α_{134}, and α_{234} are positive, and we have (17).

Set

$$a = \frac{\alpha_1}{\alpha_{12}\alpha_{13}}, \quad b = \frac{\alpha_2}{\alpha_{12}\alpha_{23}}, \quad c = \frac{\alpha_3}{\alpha_{13}\alpha_{23}}, \quad d = \frac{\alpha_{124}}{\alpha_{14}\alpha_{24}}, \quad e = \frac{\alpha_{134}}{\alpha_{14}\alpha_{34}}, \quad \text{and} \quad f = \frac{\alpha_{234}}{\alpha_{24}\alpha_{34}}.$$

Note $a, b, c, d, e, f \in (0, 1]$. If the coefficient of W in P^1 is nonnegative, then, as above, $A \geq 1$; thus this coefficient is negative, whence the discriminant of P^1 is nonpositive. This yields

$$1 - \frac{\alpha_2\alpha_{34}}{\alpha_{23}} - \frac{\alpha_3\alpha_{24}}{\alpha_{23}} \leq 2\sqrt{\frac{\alpha_2\alpha_3}{\alpha_{23}^2}(\alpha_{24}\alpha_{34} - \alpha_{234})},$$

which in the notation introduced above becomes

$$1 - bx - cy \leq 2\sqrt{bxcy(1 - f)} \leq (bx + cy)\sqrt{1 - f}.$$

Thus

$$x + y \geq [(1 + \sqrt{1 - f})\max\{b, c\}]^{-1},$$

and a similar argument using P_1 gives

$$x + y \geq [(1 + \sqrt{1 - a})\max\{d, e\}]^{-1},$$

so that

$$x + y \geq \max\{[(1 + \sqrt{1 - f})\max\{b, c\}]^{-1}, [(1 + \sqrt{1 - a})\max\{d, e\}]^{-1}\}.$$

Similar arguments using P^2, P_2, P^3, and P_3 yield

$$x + z \geq \max\{[(1 + \sqrt{1 - e})\max\{a, c\}]^{-1}, [(1 + \sqrt{1 - b})\max\{d, f\}]^{-1}\}$$

and

$$y + z \geq \max\{[(1 + \sqrt{1 - d})\max\{a, b\}]^{-1}, [(1 + \sqrt{1 - c})\max\{e, f\}]^{-1}\}.$$

In particular, we have $x + y, x + z, y + z \geq 1/2$.

The proof is now an easy consequence of

$$\inf\{[(1 + \sqrt{1 - v})u]^{-1} + [(1 + \sqrt{1 - u})v]^{-1} : 0 < u, v \leq 1\} = \frac{27}{16}, \tag{18}$$

verification of which is a straightforward calculus exercise which we omit. Assuming (18) and, without loss of generality, $a \geq b$ and $d \geq e$, we have

$$(x + y) + (y + z) \geq [(1 + \sqrt{1 - a})d]^{-1} + [(1 + \sqrt{1 - d})a]^{-1} \geq \frac{27}{16},$$

which, combined with $x + z \geq 1/2$, gives the final contradiction $2A > 2$.

■

Proof of Theorem 7. First observe that LC[m] is equivalent to having

$$\text{NC+ implies ULC for measures in } \mathcal{M}_n \tag{19}$$

for all $n \le m$. Suppose $\mu \in \mathcal{M}_n$ has rank sequence $(r_i)_{i=0}^n$. Notice that in general for (19) it is enough to show that NC+ implies

$$r_k^2 \binom{n}{k}^{-2} \ge r_{k-1} r_{k+1} \binom{n}{k-1}^{-1} \binom{n}{k+1}^{-1} \tag{20}$$

for $1 \le k \le \lfloor n/2 \rfloor$, since the measure $\mu^* \in \mathcal{M}_n$ with $\mu^*(X) = \mu([n] \setminus X)$ has rank sequence $(r_{n-i})_{i=0}^n$ and is NC+ if and only if μ is. In fact, Choe and Wagner [7] show that (20) holds for $k = 1$ and *any* n (assuming NC+). This gives (19) for $n \le 3$ (and hence LC[3]) and for the cases of interest here—that is, $n = 4, 5$—reduces the problem to proving (20) when $k = 2$.

Assume $n \in \{4, 5\}$. Using inequalities of the form

$$\alpha_i \alpha_{ijl} \le \alpha_{ij} \alpha_{il}$$

(which follow from NC+) we obtain

$$\Sigma_{1,3}^1 \le \Sigma_{2,2}^1. \tag{21}$$

It follows from Lemma 30 (for $n = 4$ this is the conclusion of the lemma, and for $n = 5$ we apply the lemma to each of the five conditional measures $\mu(\cdot | \eta_i = 0)$) that

$$3\Sigma_{1,3}^0 \le 4\Sigma_{2,2}^0. \tag{22}$$

Note also that Cauchy-Schwarz implies that the average size of a term in $\Sigma_{2,2}^2$ is at least the average size of a term in either of $\Sigma_{2,2}^0$, $\Sigma_{2,2}^1$; that is,

$$\Sigma_{2,2}^2 \ge \frac{4}{(n-2)(n-3)} \Sigma_{2,2}^0 \quad \text{and} \quad \Sigma_{2,2}^2 \ge \frac{1}{n-2} \Sigma_{2,2}^1. \tag{23}$$

Thus, finally, we have (20) for $k = 2$:

$$9r_1 r_3 = 9\Sigma_{1,3}^0 + 9\Sigma_{1,3}^1 \le 12\Sigma_{2,2}^0 + 9\Sigma_{2,2}^1 \le 8\Sigma_{2,2}^0 + 8\Sigma_{2,2}^1 + 4\Sigma_{2,2}^2 = 4r_2^2 \quad \text{if } n = 4$$

and

$$2r_1 r_3 = 2\Sigma_{1,3}^0 + 2\Sigma_{1,3}^1 \le \frac{8}{3}\Sigma_{2,2}^0 + 2\Sigma_{2,2}^1 \le 2\Sigma_{2,2}^0 + 2\Sigma_{2,2}^1 + \Sigma_{2,2}^2 = r_2^2 \quad \text{if } n = 5,$$

where in each case we used (21) and (22) for the first inequality and (23) for the second.

■

5   Urns

Finally, in this short section, we just give the easy examples justifying Proposition 17 and the remark following it. (Recall that these say that log-concavity and the Rayleigh property fail for competing urn measures (with identical balls). As mentioned in Section 1, positive results for competing urns will appear separately.) In both examples we use $p(j)$ for the probability that any given ball lands in urn j.

Example 31 *Suppose we have three balls and urns $0, \ldots, n$, with $p(0) = \varepsilon$ and $p(1) = \cdots = p(n) = (1 - \varepsilon)/n$, where ε is small and $n\varepsilon^{3/2}$ is large. Then for the associated rank sequence, say $a = (a_1, a_2, a_3)$, we have $a_1 \approx \varepsilon^3$, $a_3 \approx (1 + 2\varepsilon)(1 - \varepsilon)^2$ and*

$$a_2 = 3\varepsilon^2(1 - \varepsilon) + 3\varepsilon(1 - \varepsilon)^2/n + 3(1 - \varepsilon)^3(n - 1)/n^2 \approx 3\varepsilon^2(1 - \varepsilon);$$

so LC fails for a.

(We don't know what happens if we replace "LC" by "unimodal.")

Example 32 *Suppose we have two balls, urns $0, 1, 2$, and $p(1) = p(2) = \varepsilon$, with ε small, and impose the external field $(\varepsilon, 1, 1)$. Then for the corresponding urn measure μ on $\{0, 1\}^{\{0,1,2\}}$ (and η the random configuration) we have $\mu(\eta_1 = \eta_2 = 1) \propto 2\varepsilon^2$, $\mu(\eta_1 = \eta_2 = 0) \propto (1 - 2\varepsilon)^2\varepsilon$, and $\mu(\eta_1 = 1, \eta_2 = 0), \mu(\eta_1 = 0, \eta_2 = 1) \propto \varepsilon^2 + 2\varepsilon^2(1 - 2\varepsilon)$, so that η_1 and η_2 are strictly positively correlated.*

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